Formula of Verdermonde Determinant

Common Knowledge

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Approach Induction

Target "Kill" the last row and column.

1 First Attempt

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} = \begin{vmatrix} 0 & \lambda_1 - \lambda_n & \cdots & \lambda_1^{n-1} - \lambda_n^{n-1} \\ 0 & \lambda_2 - \lambda_n & \cdots & \lambda_2^{n-1} - \lambda_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} \quad (R_k \to R_k - R_n, k \neq n)$$

Question: Use $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$?

$$\begin{aligned} &\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} \\ &= (-1)^{n+1} \begin{vmatrix} \lambda_1 - \lambda_n & \lambda_1^2 - \lambda_n^2 & \cdots & \lambda_1^{n-1} - \lambda_n^{n-1} \\ \lambda_2 - \lambda_n & \lambda_2^2 - \lambda_n^2 & \cdots & \lambda_2^{n-1} - \lambda_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_n & \lambda_{n-1}^2 - \lambda_n^2 & \cdots & \lambda_{n-1}^{n-1} - \lambda_n^{n-1} \end{vmatrix} \\ &= (-1)^{n+1}(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)\cdots(\lambda_{n-1} - \lambda_n) \begin{vmatrix} 1 & \lambda_1 + \lambda_n & \cdots & \lambda_1^{n-2} + \lambda_1^{n-3}\lambda_n + \cdots + \lambda_1\lambda_n^{n-3} + \lambda_n^{n-2} \\ 1 & \lambda_2 + \lambda_n & \cdots & \lambda_2^{n-2} + \lambda_2^{n-3}\lambda_n + \cdots + \lambda_2\lambda_n^{n-3} + \lambda_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} + \lambda_n & \cdots & \lambda_{n-1}^{n-2} + \lambda_{n-1}^{n-3}\lambda_n + \cdots + \lambda_{n-1}\lambda_n^{n-3} + \lambda_n^{n-2} \end{vmatrix} \end{aligned}$$

It doesn't seem good. Let's try another way.

2 Second Attempt

We "kill" the last row by making the entries zero, except the first one.

$$
\begin{align}
&\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} \tag{1} \\
&= \begin{vmatrix} 1 & \lambda_1 - \lambda_n & \cdots & \lambda_1^{n-1} - \lambda_n \lambda_1^{n-2} \\ 1 & \lambda_2 - \lambda_n & \cdots & \lambda_2^{n-1} - \lambda_n \lambda_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} - \lambda_n & \cdots & \lambda_{n-1}^{n-1} - \lambda_n \lambda_2 n - 1n - 2 \\ 1 & 0 & \cdots & 0 \end{vmatrix} \quad (C_k \to C_k - \lambda_n C_{k-1}) \tag{2} \\
&= (-1)^{n+1} \begin{vmatrix} \lambda_1 - \lambda_n & \cdots & (\lambda_1 - \lambda_n)\lambda_1^{n-2} \\ \lambda_2 - \lambda_n & \cdots & (\lambda_2 - \lambda_n)\lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_n & \cdots & (\lambda_{n-1} - \lambda_n)\lambda_{n-1}^{n-2} \end{vmatrix} \tag{3} \\
&= (-1)^{n+1}(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)\cdots(\lambda_{n-1} - \lambda_n) \begin{vmatrix} 1 & \cdots & \lambda_1^{n-2} \\ 1 & \cdots & \lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \lambda_{n-1}^{n-2} \end{vmatrix} \tag{4} \\
&= (\lambda_n - \lambda_1)(\lambda_n - \lambda_2)\cdots(\lambda_n - \lambda_{n-1}) \begin{vmatrix} 1 & \cdots & \lambda_1^{n-2} \\ 1 & \cdots & \lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \lambda_{n-1}^{n-2} \end{vmatrix} \tag{5}
\end{align}
$$

- In (2), $k = n-2, n-3, \ldots, 3, 2$. Start with C_{n-2} *first*.
- In the last step, I changed the factors from $\lambda_k - \lambda_n$ to $\lambda_n - \lambda_k$, where $k = 1, 2, \ldots, n-1$, so that it is in the form of what you see in Wikipedia.

OK! I believe that you know what to do next to get the formula for Verdermonde determinant.

$$
\boxed{\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)}
$$